JOURNAL OF APPROXIMATION THEORY 38, 101–103 (1983)

Note

A Note on Rational Approximation to $(1-x)^{\alpha}$

A. McD. MERCER

Department of Mathematics and Statistics, University of Guelph, Guelph, Ontario N1G 2W1, Canada

Communicated by Oved Shisha

Received April 26, 1982

Let the set of polynomials of degree at most n and having nonnegative real coefficients be denoted by Π_n^+. If $\|f\|$ means $\sup_{[0,1]} |f(x)|$, let us write

$$H_n[f] = \inf_{p,q\in\Pi_n^+} \|f - p/q\|.$$

It has been shown recently that the exact order of $H_n[(1-x)^{1/2}]$ is $n^{-1/2}$. This is a consequence of the result

$$\|(1-x)^{1/2} - (p(x)/q(x))\| \geqslant \tfrac{1}{4} n^{-1/2} \qquad (p, q \in \Pi_n^+, n \geqslant 12) \tag{1}$$

due to Reddy [1] and the result

$$\|(1-x)^{1/2} - T_n^{-1}(x)\| \leqslant (\sqrt{\pi}/2)\, n^{-1/2} \qquad (n \geqslant 1) \tag{2}$$

due to Bundschuh [2]. Here $T_n(x)$ denotes the nth Taylor polynomial of $(1-x)^{-1/2}$ which clearly belongs to Π_n^+.

In the present note we generalize result (2) to the case of the function $(1-x)^{\alpha}$ $(0 < \alpha \leqslant 1)$. The method used is quite different from that used by Bundschuh and treats all α in the range $0 < \alpha \leqslant 1$ simultaneously. Our result is the Theorem stated below.

The proof of Reddy's result (1) can be extended with little change to cover each value of α, $0 < \alpha \leqslant 1$, whence it is found that

$$\|(1-x)^{\alpha} - (p(x)/q(x))\| \geqslant \tfrac{1}{4} n^{-\alpha} \qquad (p, q \in \Pi_n^+, n \geqslant 12).$$

This, combined with our Theorem, gives the exact order of $H_n[(1-x)^{\alpha}]$ as $n^{-\alpha}$ $(0 < \alpha \leqslant 1)$.

Our main result is

0021-9045/83 $3.00

THEOREM. *If* $T_n(\alpha, x)$ *is the nth Taylor polynomial of* $(1-x)^{-\alpha}$ $(0<\alpha\leqslant 1)$ *(which belongs to* Π_n^+*), then*

$$\|(1-x)^\alpha - T_n^{-1}(\alpha, x)\| \leqslant K\Gamma(\alpha)\, n^{-\alpha} \qquad (n \geqslant 1),$$

where K is a constant independent of both n and α.

Proof. The function $(1-x)^{-\alpha}$ is the unique solution of the differential equation

$$(1-x)\,y' - \alpha y = 0 \qquad \text{with} \quad y(0) = 1.$$

Written in series form the solution is

$$(1-x)^{-\alpha} = \sum_{k=0}^{\infty} \frac{\Gamma(k+\alpha)}{k!\,\Gamma(\alpha)} x^k \qquad (|x|<1)$$

and so

$$T_n(\alpha, x) = \sum_{k=0}^{n} \frac{\Gamma(k+\alpha)}{k!\,\Gamma(\alpha)} x^k.$$

It is then easy to see that $T_n(\alpha, x)$ is the unique solution of the differential equation

$$(1-x)\,y' - \alpha y = -\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha)\,n!} x^n \qquad \text{with} \quad y(0) = 1.$$

Solving this by a standard technique we find that

$$T_n(\alpha, x) = (1-x)^{-\alpha} - \frac{\Gamma(n+\alpha+1)}{n!\,\Gamma(\alpha)} (1-x)^{-\alpha} \int_0^x t^n (1-t)^{\alpha-1}\, dt.$$

Dividing by $(1-x)^{-\alpha}\, T_n(\alpha, x)$, we get

$$(1-x)^\alpha = \frac{1}{T_n(\alpha, x)} - \frac{\Gamma(n+\alpha+1)}{n!\,\Gamma(\alpha)} \frac{1}{T_n(\alpha, x)} \int_0^x t^n (1-t)^{\alpha-1}\, dt. \qquad (3)$$

We now examine the last term. We have

$$\begin{aligned} 0 \leqslant \int_0^x t^n (1-t)^{\alpha-1}\, dt &= x^{n+1} \int_0^1 t^n (1-xt)^{\alpha-1}\, dt \\ &\leqslant x^{n+1} \int_0^1 t^n (1-t)^{\alpha-1}\, dt \\ &= x^{n+1} \frac{\Gamma(n+1)\,\Gamma(\alpha)}{\Gamma(n+\alpha+1)}. \end{aligned}$$

Also

$$T_n(\alpha, x) = \sum_{k=0}^{n} \frac{\Gamma(k+\alpha)}{\Gamma(\alpha)} \frac{1}{k!} x^k \geqslant \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\, n!} \sum_{k=0}^{n} x^k$$

because the coefficients are positive and decrease with k. We note that

$$\frac{x^{n+1}}{T_n(\alpha, x)} \leqslant \frac{x^{n+1}}{\sum_{k=0}^{n} x^k} \frac{\Gamma(\alpha)\, n!}{\Gamma(n+\alpha)} \leqslant \frac{1}{n+1} \frac{\Gamma(\alpha)\, n!}{\Gamma(n+\alpha)}.$$

The last step here depends on the observation that $x^{n+1}(\sum_{k=0}^{n} x^k)^{-1}$ is an increasing function in $0 \leqslant x \leqslant 1$. Applying these results to (3), we now see that

$$0 \leqslant \frac{1}{T_n(\alpha, x)} - (1-x)^\alpha \leqslant \frac{n+\alpha}{n+1} \frac{\Gamma(\alpha)\, \Gamma(n+1)}{\Gamma(n+\alpha+1)}.$$

Since $n^\alpha(\Gamma(n)/\Gamma(n+\alpha)) \to 1$ as $n \to \infty$, uniformly with respect to α in $0 < \alpha \leqslant 1$, the proof of the theorem is complete.

References

1. A. R. Reddy, A note on rational approximation to $(1-x)^{1/2}$, *J. Approx. Theory* **25** (1979), 31–33.
2. P. Bundschuh, A remark on Reddy's paper on the rational approximation of $(1-x)^{1/2}$, *J. Approx. Theory* **32** (1981), 167–169.

Printed in Belgium